# Similarity Search

## The String Edit Distance

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## Outline







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## Motivation

- How different are
  - `hello` and `hello`?
  - `hello` and `hallo`?
  - `hello` and `hell`?
  - `hello` and `shell`?

## What is a String Distance Function?

**Definition (String Distance Function)**

Given a finite alphabet $\Sigma$, a *string distance function*, $\delta_s$, maps each pair of strings $(x, y) \in \Sigma^* \times \Sigma^*$ to a positive real number (including zero).

$$\delta_s : \Sigma^* \times \Sigma^* \to \mathbb{R}_0^+$$

- $\Sigma^*$ is the set of all strings over $\Sigma$, including the empty string $\varepsilon$.





## The String Edit Distance

**Definition (String Edit Distance)**

The *string edit distance* between two strings, $\text{ed}(x, y)$, is the minimum number of character insertions, deletions and replacements that transforms $x$ to $y$.

- Example:
  - `hello`→`hallo`: replace `e` by `a`
  - `hello`→`hell`: delete `o`
  - `hello`→`shell`: delete `o`, insert `s`
- Also called *Levenshtein distance*.[1]

[1] Levenshtein introduced this distance for signal processing in 1965 [Lev65].





## Outline







## Gap Representation

- Gap representation of the string transformation $x \to y$:
  Place string $x$ above string $y$
  - with a gap in $x$ for every insertion,
  - with a gap in $y$ for every deletion,
  - with different characters in $x$ and $y$ for every replacement.
- Any sequence of edit operations can be represented with gaps.
- Example:

```
  h a l l o
s h e l l
```

  - insert `s`
  - replace `a` by `e`
  - delete `o`

## Deriving the Recursive Formula

- Example:

```
  h a l l o
s h e l l
```

- Given: Gap representation, gap$(x, y)$, of the shortest edit distance between two strings $x$ and $y$, such that gap$(x, y) = $ ed$(x, y)$.
- Claim:
  - If we remove the last column,
  - then the remaining columns represent the shortest edit distance, gap$(x', y') = $ ed$(x', y')$, between the remaining substrings, $x'$ and $y'$.
- Proof (by contradiction):
  - Last column contributes with $c = 0$ or $c = 1$ to gap$(x, y)$, thus $gap(x, y) = gap(x', y') + c$.
  - If we assume ed$(x', y') <$ gap$(x', y')$, then we could find a new gap representation gap$^*(x', y') = $ ed$(x', y') < gap(x', y')$ such that gap$^*(x, y) = $ gap$^*(x', y') + c <$ gap$(x', y') + c = $ ed$(x, y)$. □





## Deriving the Recursive Formula

- Example:

```
  h a l l o
s h e l l
```

- Notation:
  - $x[1 \dots i]$ is the substring of the first $i$ characters of $x$ ($x[1 \dots 0] = \varepsilon$)
  - $x[i]$ is the $i$-th character of $x$
- Recursive Formula:

$$
\begin{aligned}
\text{ed}(\varepsilon, \varepsilon) &= 0 \\
\text{ed}(x[1..i], \varepsilon] &= i \\
\text{ed}(\varepsilon, y[1..j] &= j \\
\text{ed}(x[1..i], y[1..j]) &= \min(\text{ed}(x[1..i-1], y[1..j-1]) + c, \\
&\qquad \text{ed}(x[1..i-1], y[1..j]) + 1, \\
&\qquad \text{ed}(x[1..i], y[1..j-1]) + 1)
\end{aligned}
$$

where $c = 0$ if $x[i] = y[j]$, otherwise $c = 1$.





## Brute Force Algorithm

**ed-bf$(x, y)$**

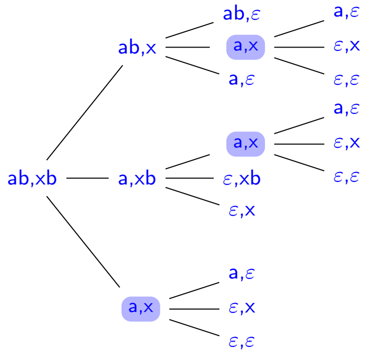
$m = |x|$, $n = |y|$
**if** $m = 0$ **then return** $n$
**if** $n = 0$ **then return** $m$
**if** $x[m] = y[n]$ **then** $c = 0$ **else** $c = 1$
**return** min(ed-bf$(x, y[1 \dots n-1]) + 1$,
ed-bf$(x[1 \dots m-1], y) + 1$,
ed-bf$(x[1 \dots m-1], y[1 \dots n-1]) + c$)





## Brute Force Algorithm

- Recursion tree for ed-bf(ab, xb):




- Exponential runtime in string length :-(
- Observation: Subproblems are computed repeatedly (e.g. ed-bf(a, x) is computed 3 times)
- Approach: Reuse previously computed results!

# Outline







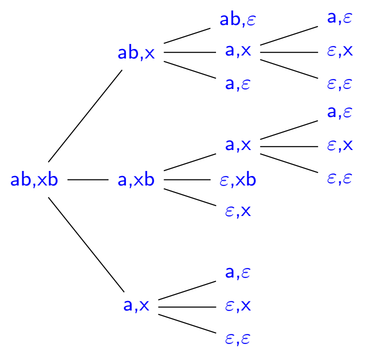
# Dynamic Programming Algorithm

- Store distances between all prefixes of $x$ and $y$
- Use matrix $C_{0..m,0..n}$ with

$$C_{i,j} = \text{ed}(x[1\ldots i], y[1\ldots j])$$

  where $x[1..0] = y[1..0] = \varepsilon$.
- Example:

| | $\varepsilon$ | x | b |
|---|---|---|---|
| $\varepsilon$ | 0 | 1 | 2 |
| a | 1 | 1 | 2 |
| b | 2 | 2 | **1** |








# Dynamic Programming Algorithm

**ed-dyn($x, y$)**

$C$ : *array*$[0..|x|][0..|y|]$
**for** $i = 0$ **to** $|x|$ **do** $C[i,0] = i$
**for** $j = 1$ **to** $|y|$ **do** $C[0,j] = j$
**for** $j = 1$ **to** $|y|$ **do**
    **for** $i = 1$ **to** $|x|$ **do**
        **if** $x[i] = y[j]$ **then** $c = 0$ **else** $c = 1$
        $C[i,j] = \min(C[i-1,j-1] + c,$
                $C[i-1,j] + 1,$
                $C[i,j-1] + 1)$





# Understanding the Solution

- Example:

$x$ = moon
$y$ = mond

ins →, del ↓

| | $\varepsilon$ | m | o | n | d |
|---|---|---|---|---|---|
| $\varepsilon$ | 0 | 1 | 2 | 3 | 4 |
| m | 1 | 0 | 1 | 2 | 3 |
| o | 2 | 1 | 0 | 1 | 2 |
| o | 3 | 2 | 1 | 1 | 2 |
| n | 4 | 3 | 2 | 1 | 2 |

```
m o o n
m o n d

m o o n
m   o n d

m o o n
m o   n d
```

- Solution 1: replace n by d and (second) o by n in $x$
- Solution 2: insert d after n and delete (first) o in $x$
- Solution 3: insert d after n and delete (second) o in $x$

## Dynamic Programming Algorithm – Properties

- Complexity:
  - $O(mn)$ time (nested for-loop)
  - $O(mn)$ space (the $(m+1)\times(n+1)$-matrix $C$)
- Improving space complexity (assume $m < n$):
  - we need only the previous column to compute the next column
  - we can forget all other columns
  - $\Rightarrow O(m)$ space complexity





## Dynamic Programming Algorithm

**ed-dyn$^+(x, y)$**

$col_0 : array[0..|x|]$
$col_1 : array[0..|x|]$
**for** $i = 0$ **to** $|x|$ **do** $col_0[i] = i$
**for** $j = 1$ **to** $|y|$ **do**
  $col_1[0] = j$
  **for** $i = 1$ **to** $|x|$ **do**
    **if** $x[i] = y[j]$ **then** $c = 0$ **else** $c = 1$
    $col_1[i] = \min(col_0[i-1] + c,$
          $col_1[i-1] + 1,$
          $col_0[i] + 1)$
  $col_0 = col1$





## Outline







## Distance Metric

**Definition (Distance Metric)**

A distance function $\delta$ is a *distance metric* if and only if for any $x, y, z$ the following hold:

- $\delta(x, y) = 0 \Leftrightarrow x = y$ (identity)
- $\delta(x, y) = \delta(y, x)$ (symmetric)
- $\delta(x, y) + \delta(y, z) \geq \delta(x, z)$ (triangle inequality)

Examples:

- the Euclidean distance is a metric
- $d(a, b) = a - b$ is not a metric (not symmetric)

## Introducing Weights

- Look at the edit operations as a set of rules with a cost:

$$
\begin{aligned}
\alpha(\varepsilon, b) &= \omega_{ins} && \text{(insert)} \\
\alpha(a, \varepsilon) &= \omega_{del} && \text{(delete)} \\
\alpha(a, b) &= \begin{cases} \omega_{rep} & \text{if } a \neq b \\ 0 & \text{if } a = b \end{cases} && \text{(replace)}
\end{aligned}
$$

  where $a, b \in \Sigma$, and $\omega_{ins}, \omega_{del}, \omega_{rep} \in \mathbb{R}_0^+$.
- Edit script: sequence of rules that transform $x$ to $y$
- Edit distance: edit script with minimum cost (adding up costs of single rules)
- Example: so far we assumed $\omega_{ins} = \omega_{del} = \omega_{rep} = 1$.





## Weighted Edit Distance

- Recursive formula with weights:

$$
\begin{aligned}
C_{0,0} &= 0 \\
C_{i,j} &= \min(C_{i-1,j-1} + \alpha(x[i], y[j]), \\
&\qquad C_{i-1,j} + \alpha(x[i], \varepsilon), \\
&\qquad C_{i,j-1} + \alpha(\varepsilon, y[j]))
\end{aligned}
$$

  where $\alpha(a, a) = 0$ for all $a \in \Sigma$, and $C_{-1,j} = C_{i,-1} = \infty$.
- We can easily adapt the dynamic programming algorithm.





## Variants of the Edit Distance

- Unit cost edit distance (what we did so far):
  - $\omega_{ins} = \omega_{del} = \omega_{rep} = 1$
  - $0 \leq ed(x, y) \leq \max(|x|, |y|)$
  - distance metric
- Hamming distance [Ham50, SK83]:
  - called also "string matching with $k$ mismatches"
  - allows only replacements
  - $\omega_{rep} = 1$, $\omega_{ins} = \omega_{del} = \infty$
  - $0 \leq d(x, y) \leq |x|$ if $|x| = |y|$, otherwise $d(x, y) = \infty$
  - distance metric
- Longest Common Subsequence (LCS) distance [NW70, AG87]:
  - allows only insertions and deletions
  - $\omega_{ins} = \omega_{del} = 1$, $\omega_{rep} = \infty$
  - $0 \leq d(x, y) \leq |x| + |y|$
  - distance metric
  - $LCS(x, y) = (|x| + |y| - d(x, y))/2$





## Allowing Transposition

- Transpositions
  - switch two adjacent characters
  - can be simulated by delete and insert
  - typos are often transpositions
- New rule for transposition

$$\alpha(ab, ba) = \omega_{trans}$$

  allows us to assign a weight different from $\omega_{ins} + \omega_{del}$
- Recursive formula that includes transposition:

$$
\begin{aligned}
C_{0,0} &= 0 \\
C_{i,j} &= \min(C_{i-1,j-1} + \alpha(x[i], y[j]), \\
&\qquad C_{i-1,j} + \alpha(x[i], \varepsilon), \\
&\qquad C_{i,j-1} + \alpha(\varepsilon, y[j]), \\
&\qquad C_{i-2,j-2} + \alpha(x[i-1]x[i], y[j-1]y[j]))
\end{aligned}
$$

  where $\alpha(ab, cd) = \infty$ if $a \neq d$ or $b \neq c$, $\alpha(a, a) = 0$ for all $a \in \Sigma$, and $C_{-1,j} = C_{i,-1} = C_{-2,j} = C_{j,-2} = \infty$.

## Example: Edit Distance with Transposition

- Example: Compute distance between $x =$meal and $y =$mael using the edit distance with transposition ($\omega_{ins} = \omega_{del} = \omega_{rep} = \omega_{trans} = 1$)

| | ε | m | a | e | l |
|---|---|---|---|---|---|
| ε | 0 | 1 | 2 | 3 | 4 |
| m | 1 | 0 | 1 | 2 | 3 |
| e | 2 | 1 | 1 | 1 | 2 |
| a | 3 | 2 | 1 | **1** | 2 |
| l | 4 | 3 | 2 | 2 | 1 |

- The value in red results from the transposition of ea to ae.





## Text Searching

- Goal:
  - search pattern $p$ in text $t$ ($|p| < |t|$)
  - allow $k$ errors
  - match may start at any position of the text
- Difference to distance computation:
  - $C_{0,j} = 0$ (instead of $C_{0,j} = j$, as text may start at any position)
  - result: all $C_{m,j} \leq k$ are endpoints of matches





## Example: Text Searching

- Example:

$p$ = survey
$t$ = surgery
$k$ = 2

| | ε | s | u | r | g | e | r | y |
|---|---|---|---|---|---|---|---|---|
| ε | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| s | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| u | 2 | 1 | 0 | 1 | 2 | 2 | 2 | 2 |
| r | 3 | 2 | 1 | 0 | 1 | 2 | 2 | 3 |
| v | 4 | 3 | 2 | 1 | 1 | 2 | 3 | 3 |
| e | 5 | 4 | 3 | 2 | 2 | 1 | 2 | 3 |
| y | 6 | 5 | 4 | 3 | 3 | **2** | **2** | **2** |

- Solutions: 3 matching positions with $k \leq 2$ found.

```
s u r v e y
s u r g e

s u r v e y
s u r g e r

s u r v e   y
s u r g e r y
```



Alberto Apostolico and Zvi Galill.
The longest common subsequence problem revisited.
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