Similarity Search
The String Edit Distance

Nikolaus Augsten
nikolaus.augsten@plus.ac.at
Department of Computer Science
University of Salzburg

https://dbresearch.uni-salzburg.at

WS 2023/24

Version October 20, 2023

Outline

Outline

Motivation

- How different are
  - `hello` and `hello`?
  - `hello` and `hallo`?
  - `hello` and `hell`?
  - `hello` and `shell`?

What is a String Distance Function?

Definition (String Distance Function)

Given a finite alphabet Σ, a *string distance function*, δ_s, maps each pair of strings $(x, y) \in \Sigma^* \times \Sigma^*$ to a positive real number (including zero).

$$\delta_s : \Sigma^* \times \Sigma^* \to \mathbb{R}_0^+$$

- Σ^* is the set of all strings over Σ, including the empty string ε.

The String Edit Distance

Definition (String Edit Distance)

The *string edit distance* between two strings, $\mathrm{ed}(x, y)$, is the minimum number of character insertions, deletions and replacements that transforms x to y.

- Example:
  - `hello`→`hallo`: replace `e` by `a`
  - `hello`→`hell`: delete `o`
  - `hello`→`shell`: delete `o`, insert `s`
- Also called *Levenshtein distance*.[1]

[1] Levenshtein introduced this distance for signal processing in 1965 [Lev65].

Outline

Gap Representation

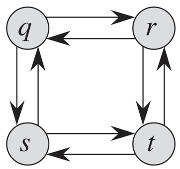
- Gap representation of the string transformation $x \to y$:
  Place string x above string y
  - with a gap in x for every insertion,
  - with a gap in y for every deletion,
  - with different characters in x and y for every replacement.
- Any sequence of edit operations can be represented with gaps.
- Example:

```
  h a l l o
s h e l l
```

  - insert `s`
  - replace `a` by `e`
  - delete `o`

Deriving the Recursive Formula: Optimal Substructure

- Recursive solution: is applicable only to problems with optimal substructure property.
- Optimal substructure property of a problem:
  - optimal solution to larger problem computable from the optimal solutions of subproblems
- Examples:
  - Shortest path has optimal substructure: If a is on shortest path P from a to b $\Rightarrow$ the section $a \to c$ on P is the shortest path between a and c.
  - Longest simple[2] path does *not* have optimal substructure. Counter example [CLRS09]: Consider longest path $q \to t$ and subpath $q \to r$.



[2] i.e., the path has no cycles

Deriving the Recursive Formula: Optimal Substructure

Lemma (Optimal Substructure of String Edit Distance Problem)

Given a gap representation, gap(x, y)*, between two strings* x *and* y*, such that the cost of* gap(x, y) *is the string edit distance* ed(x, y)*.*
If we remove the last column of gap(x, y)*, then the gap representation of the remaining columns,* gap(x', y')*, has cost* ed(x', y') *between the resulting substrings,* x' *and* y'*.*

```
  h a l l|o
s h e l l|
```

- Example:
  - $x =$ `hallo`, $y =$ `shell`, $cost(\text{gap}(x, y)) = \text{ed}(x, y) = 3$
  - $x' =$ `hall`, $y =$ `shell`, $cost(\text{gap}(x', y')) = \text{ed}(x', y') = 2$

Deriving the Recursive Formula: Optimal Substructure

Proof: Optimal Substructure String Edit Distance (by contradiction).

- Last column contributes with $c = 0$ or $c = 1$ to cost of gap(x, y), thus:

$$cost(\text{gap}(x, y)) = cost(\text{gap}(x', y')) + c$$

- Assume gap(x', y') is not optimal, i.e., $cost(\text{gap}(x', y')) > \text{ed}(x', y')$. Let gap$^*(x', y')$ be the respective gap representation:

$$cost(\text{gap}^*(x', y')) = \text{ed}(x', y') < cost(\text{gap}(x', y'))$$

- By extending gap$^*(x', y')$ with the last column, we get a gap representation gap$^*(x, y)$ with cost below ed(x, y), which contradicts the definition of the edit distance.

$$\begin{aligned} cost(\text{gap}^*(x, y)) &= cost(\text{gap}^*(x', y')) + c \\ &< cost(\text{gap}(x', y')) + c = \text{ed}(x, y) \end{aligned}$$

□

Deriving the Recursive Formula

- Example:

```
  h a l l o
s h e l l
```

- Notation:
  - $x[1 \dots i]$ is the substring of the first i characters of x ($x[1 \dots 0] = \varepsilon$)
  - $x[i]$ is the i-th character of x
- Recursive Formula:

$$\begin{aligned} \text{ed}(\varepsilon, \varepsilon) &= 0 \\ \text{ed}(x[1..i], \varepsilon]) &= i \\ \text{ed}(\varepsilon, y[1..j]) &= j \\ \text{ed}(x[1..i], y[1..j]) &= \min(\text{ed}(x[1..i-1], y[1..j-1]) + c, \\ &\qquad\quad \text{ed}(x[1..i-1], y[1..j]) + 1, \\ &\qquad\quad \text{ed}(x[1..i], y[1..j-1]) + 1) \end{aligned}$$

where $c = 0$ if $x[i] = y[j]$, otherwise $c = 1$.

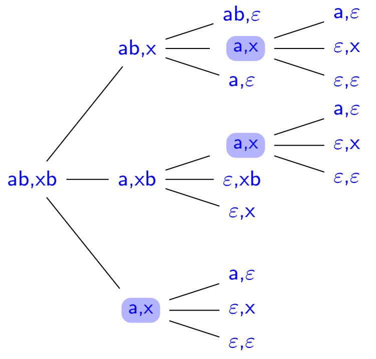
Brute Force Algorithm

ed-bf(x, y)

$m = |x|$, $n = |y|$
if $m = 0$ **then return** n
if $n = 0$ **then return** m
if $x[m] = y[n]$ **then** $c = 0$ **else** $c = 1$
return $\min(\text{ed-bf}(x, y[1 \dots n-1]) + 1,$
$\quad \text{ed-bf}(x[1 \dots m-1], y) + 1,$
$\quad \text{ed-bf}(x[1 \dots m-1], y[1 \dots n-1]) + c)$

Brute Force Algorithm

- Recursion tree for ed-bf(ab, xb):



- Exponential runtime in string length :-(
- Observation: Subproblems are computed repeatedly (e.g. ed-bf(a, x) is computed 3 times)
- Approach: Reuse previously computed results!

Outline

Dynamic Programming Algorithm – Top Down

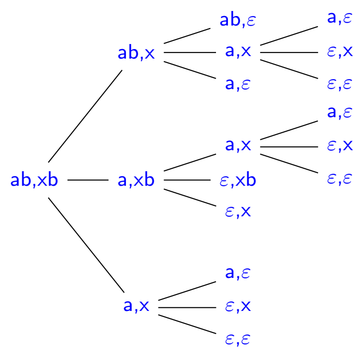
- Store distances between all prefixes of x and y
- Use matrix $C_{0..m,0..n}$ with

$$C_{i,j} = \text{ed}(x[1 \dots i], y[1 \dots j])$$

where $x[1..0] = y[1..0] = \varepsilon$.

- Example:

	ε	x	b
ε	0	1	2
a	1	1	2
b	2	2	**1**



Dynamic Programming Algorithm – Bottom Up

ed-dyn(x, y)

C : $array[0..|x|][0..|y|]$
for $i = 0$ **to** $|x|$ **do** $C[i, 0] = i$
for $j = 1$ **to** $|y|$ **do** $C[0, j] = j$
for $j = 1$ **to** $|y|$ **do**
  for $i = 1$ **to** $|x|$ **do**
    if $x[i] = y[j]$ **then** $c = 0$ **else** $c = 1$
    $C[i, j] = \min(C[i-1, j-1] + c,$
          $C[i-1, j] + 1,$
          $C[i, j-1] + 1)$

Dynamic Programming Algorithm – Properties

- Complexity:
  - $O(mn)$ time (nested for-loop)
  - $O(mn)$ space (the $(m+1)\times(n+1)$-matrix C)
- Improving space complexity (assume $m < n$):
  - we need only the previous column to compute the next column
  - we can forget all other columns
  - $\Rightarrow O(m)$ space complexity

Understanding the Solution

- Example:

x = `moon`
y = `mond`

ins →

	ren ↘	ε	m	o	n	d
	ε	0	←1	←2	←3	←4
del	m	1	0	←1	←2	←3
↓	o	2	1	0	←1	←2
	o	3	2	1	1	←2
	n	4	3	2	1	←2

- Each arrow represents an edit operation with minimal cost
- Cost 2 in cell (`n`, `d`) can either result from replacing `n` by `d` (diagonal arrow) or by inserting `d` (horizontal arrow)
- Each path from bottom right to top left corner represents a valid set of edit operations

Understanding the Solution

- Example:

x = `moon`
y = `mond`

ins →

		ε	m	o	n	d
	ε	0	1	2	3	4
del ↓	m	1	0	1	2	3
	o	2	1	0	1	2
	o	3	2	1	1	2
	n	4	3	2	1	←2

```
m o o n
m o n d

m o o n
m   o n d

m o o n
m o   n d
```

- Solution 1: replace `n` by `d` and (second) `o` by `n` in x
- Solution 2: insert `d` after `n` and delete (first) `o` in x
- Solution 3: insert `d` after `n` and delete (second) `o` in x

Dynamic Programming Algorithm

ed-dyn$^+(x, y)$

col_0 : $array[0..|x|]$
col_1 : $array[0..|x|]$
for $i = 0$ **to** $|x|$ **do** $col_0[i] = i$
for $j = 1$ **to** $|y|$ **do**
    $col_1[0] = j$
    for $i = 1$ **to** $|x|$ **do**
        if $x[i] = y[j]$ **then** $c = 0$ **else** $c = 1$
        $col_1[i] = \min(col_0[i-1] + c,$
                $col_1[i-1] + 1,$
                $col_0[i] + 1)$
    $col_0 = col_1$

Outline

Distance Metric

Definition (Distance Metric)

A distance function δ is a *distance metric* if and only if for any x, y, z the following hold:

- $\delta(x, y) = 0 \Leftrightarrow x = y$ (identity)
- $\delta(x, y) = \delta(y, x)$ (symmetric)
- $\delta(x, y) + \delta(y, z) \geq \delta(x, z)$ (triangle inequality)

Examples:

- the Euclidean distance is a metric
- $d(a, b) = a - b$ is not a metric (not symmetric)

Introducing Weights

- Look at the edit operations as a set of rules with a cost:

$$\begin{aligned} \alpha(\varepsilon, b) &= \omega_{ins} && \text{(insert)} \\ \alpha(a, \varepsilon) &= \omega_{del} && \text{(delete)} \\ \alpha(a, b) &= \begin{cases} \omega_{rep} & \text{if } a \neq b \\ 0 & \text{if } a = b \end{cases} && \text{(replace)} \end{aligned}$$

  where $a, b \in \Sigma$, and $\omega_{ins}, \omega_{del}, \omega_{rep} \in \mathbb{R}_0^+$.
- Edit script: sequence of rules that transform x to y
- Edit distance: edit script with minimum cost (adding up costs of single rules)
- Example: so far we assumed $\omega_{ins} = \omega_{del} = \omega_{rep} = 1$.

Weighted Edit Distance

- Recursive formula with weights:

$$\begin{aligned} C_{0,0} &= 0 \\ C_{i,j} &= \min(C_{i-1,j-1} + \alpha(x[i], y[j]), \\ &\qquad C_{i-1,j} + \alpha(x[i], \varepsilon), \\ &\qquad C_{i,j-1} + \alpha(\varepsilon, y[j])) \end{aligned}$$

  where $\alpha(a, a) = 0$ for all $a \in \Sigma$, and $C_{-1,j} = C_{i,-1} = \infty$.
- We can easily adapt the dynamic programming algorithm.

Variants of the Edit Distance

- Unit cost edit distance (what we did so far):
  - $\omega_{ins} = \omega_{del} = \omega_{rep} = 1$
  - $0 \leq ed(x, y) \leq \max(|x|, |y|)$
  - distance metric
- Hamming distance [Ham50, SK83]:
  - called also "string matching with k mismatches"
  - allows only replacements
  - $\omega_{rep} = 1$, $\omega_{ins} = \omega_{del} = \infty$
  - $0 \leq d(x, y) \leq |x|$ if $|x| = |y|$, otherwise $d(x, y) = \infty$
  - distance metric
- Longest Common Subsequence (LCS) distance [NW70, AG87]:
  - allows only insertions and deletions
  - $\omega_{ins} = \omega_{del} = 1$, $\omega_{rep} = \infty$
  - $0 \leq d(x, y) \leq |x| + |y|$
  - distance metric
  - $LCS(x, y) = (|x| + |y| - d(x, y))/2$

Allowing Transposition

- Transpositions
  - switch two adjacent characters
  - can be simulated by delete and insert
  - typos are often transpositions
- New rule for transposition

$$\alpha(ab, ba) = \omega_{trans}$$

  allows us to assign a weight different from $\omega_{ins} + \omega_{del}$
- Recursive formula that includes transposition:

$$\begin{aligned} C_{0,0} &= 0 \\ C_{i,j} &= \min(C_{i-1,j-1} + \alpha(x[i], y[j]), \\ &\qquad C_{i-1,j} + \alpha(x[i], \varepsilon), \\ &\qquad C_{i,j-1} + \alpha(\varepsilon, y[j]), \\ &\qquad C_{i-2,j-2} + \alpha(x[i-1]x[i], y[j-1]y[j])) \end{aligned}$$

  where $\alpha(ab, cd) = \infty$ if $a \neq d$ or $b \neq c$, $\alpha(a, a) = 0$ for all $a \in \Sigma$, and $C_{-1,j} = C_{i,-1} = C_{-2,j} = C_{i,-2} = \infty$.

Example: Edit Distance with Transposition

- Example: Compute distance between $x =$`meal` and $y =$`mael` using the edit distance with transposition ($\omega_{ins} = \omega_{del} = \omega_{rep} = \omega_{trans} = 1$)

	ε	m	a	e	l
ε	0	1	2	3	4
m	1	0	1	2	3
e	2	1	1	1	2
a	3	2	1	1	2
l	4	3	2	2	1

- The value in red results from the transposition of `ea` to `ae`.

Text Searching

- Goal:
  - search pattern p in text t ($|p| < |t|$)
  - allow k errors
  - match may start at any position of the text
- Difference to distance computation:
  - $C_{0,j} = 0$ (instead of $C_{0,j} = j$, as text may start at any position)
  - result: all $C_{m,j} \leq k$ are endpoints of matches

Example: Text Searching

- Example:

p = `survey`
t = `surgery`
k = 2

	ε	s	u	r	g	e	r	y
ε	0	0	0	0	0	0	0	0
s	1	0	1	1	1	1	1	1
u	2	1	0	1	2	2	2	2
r	3	2	1	0	1	2	2	3
v	4	3	2	1	1	2	3	3
e	5	4	3	2	2	1	2	3
y	6	5	4	3	3	**2**	**2**	**2**

- Solutions: 3 matching positions with $k \leq 2$ found.

```
s u r v e y
s u r g e

s u r v e y
s u r g e r

s u r v e   y
s u r g e r y
```


Summary

- Edit distance between two strings: the minimum number of edit operations that transforms one string into the another
- Dynamic programming algorithm with $O(mn)$ time and $O(m)$ space complexity, where $m \leq n$ are the string lengths.
- Basic algorithm can easily be extended in order to:
  - weight edit operations differently,
  - support transposition,
  - simulate Hamming distance and LCS,
  - search pattern in text with k errors.

Alberto Apostolico and Zvi Galill.
The longest common subsequence problem revisited.
Algorithmica, 2(1):315–336, March 1987.

Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein.
Introduction to Algorithms, 3rd Edition.
MIT Press, 2009.

Richard W. Hamming.
Error detecting and error correcting codes.
Bell System Technical Journal, 26(2):147–160, 1950.

Vladimir I. Levenshtein.
Binary codes capable of correcting spurious insertions and deletions of ones.
Problems of Information Transmission, 1:8–17, 1965.

Saul B. Needleman and Christian D. Wunsch.

A general method applicable to the search for similarities in the amino acid sequence of two proteins.
Journal of Molecular Biology, 48:443–453, 1970.

David Sankoff and Josef B. Kruskal, editors.
Time Warps, String Edits, and Macromolecules: The Theory and Practice of Sequence Comparison.
Addison-Wesley, Reading, MA, 1983.