Similarity Search

The String Edit Distance

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Outline

1. String Edit Distance
   - Motivation and Definition
   - Brute Force Algorithm
   - Dynamic Programming Algorithm
   - Edit Distance Variants

2. Conclusion

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Motivation

- How different are
  - `hello` and `hello`?
  - `hello` and `hallo`?
  - `hello` and `hell`?
  - `hello` and `shell`?

What is a String Distance Function?

Definition (String Distance Function)

Given a finite alphabet Σ, a *string distance function*, δ_s, maps each pair of strings $(x, y) \in \Sigma^* \times \Sigma^*$ to a positive real number (including zero).

$$\delta_s : \Sigma^* \times \Sigma^* \to \mathbb{R}_0^+$$

- Σ^* is the set of all strings over Σ, including the empty string ε.

The String Edit Distance

Definition (String Edit Distance)

The *string edit distance* between two strings, $\text{ed}(x, y)$, is the minimum number of character insertions, deletions and replacements that transforms x to y.

- Example:
  - `hello`→`hallo`: replace `e` by `a`
  - `hello`→`hell`: delete `o`
  - `hello`→`shell`: delete `o`, insert `s`
- Also called *Levenshtein distance.*[1]

[1] Levenshtein introduced this distance for signal processing in 1965 [Lev65].

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Gap Representation

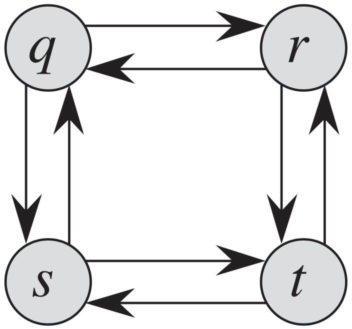
- Gap representation of the string transformation $x \rightarrow y$:
  Place string x above string y
  - with a gap in x for every insertion,
  - with a gap in y for every deletion,
  - with different characters in x and y for every replacement.
- Any sequence of edit operations can be represented with gaps.
- Example:

```
  h a l l o
s h e l l
```

  - insert `s`
  - replace `a` by `e`
  - delete `o`

Deriving the Recursive Formula: Optimal Substructure

- Recursive solution: is applicable only to problems with optimal substructure property.
- Optimal substructure property of a problem:
  - optimal solution to larger problem computable from the optimal solutions of subproblems
- Examples:
  - Shortest path has optimal substructure: If a is on shortest path P from a to $b \Rightarrow$ the section $a \rightarrow c$ on P is the shortest path between a and c.
  - Longest simple[2] path does *not* have optimal substructure. Counter example [CLRS09]: Consider longest path $q \rightarrow t$ and subpath $q \rightarrow r$.



[2] i.e., the path has no cycles

Deriving the Recursive Formula: Optimal Substructure

Lemma (Optimal Substructure of String Edit Distance Problem)

Given a gap representation, gap(x, y)*, between two strings* x *and* y*, such that the cost of* gap(x, y) *is the string edit distance* ed(x, y)*.*
If we remove the last column of gap(x, y)*, then the gap representation of the remaining columns,* gap(x', y')*, has cost* ed(x', y') *between the resulting substrings,* x' *and* y'*.*

```
  h a l l|o
s h e l l|
```

- Example:
  - $x =$ `hallo`, $y =$ `shell`, $cost(\text{gap}(x, y)) = \text{ed}(x, y) = 3$
  - $x' =$ `hall`, $y =$ `shell`, $cost(\text{gap}(x', y')) = \text{ed}(x', y') = 2$

Deriving the Recursive Formula: Optimal Substructure

Proof: Optimal Substructure String Edit Distance (by contradiction).

- Last column contributes with $c = 0$ or $c = 1$ to cost of $\text{gap}(x, y)$, thus:

$$cost(\text{gap}(x, y)) = cost(\text{gap}(x', y')) + c$$

- Assume $\text{gap}(x', y')$ is not optimal, i.e., $cost(\text{gap}(x', y')) > \text{ed}(x', y')$. Let $\text{gap}^*(x', y')$ be the respective gap representation:

$$cost(\text{gap}^*(x', y')) = \text{ed}(x', y') < cost(\text{gap}(x', y'))$$

- By extending $\text{gap}^*(x', y')$ with the last column, we get a gap representation $\text{gap}^*(x, y)$ with cost below $\text{ed}(x, y)$, which contradicts the definition of the edit distance.

$$\begin{aligned} cost(\text{gap}^*(x, y)) &= cost(\text{gap}^*(x', y')) + c \\ &< cost(\text{gap}(x', y')) + c = \text{ed}(x, y) \end{aligned}$$

□

Deriving the Recursive Formula

- Example:

```
  h a l l o
s h e l l
```

- Notation:
  - $x[1 \dots i]$ is the substring of the first i characters of x ($x[1 \dots 0] = \varepsilon$)
  - $x[i]$ is the i-th character of x
- Recursive Formula:

$$
\begin{aligned}
\mathrm{ed}(\varepsilon, \varepsilon) &= 0 \\
\mathrm{ed}(x[1..i], \varepsilon]) &= i \\
\mathrm{ed}(\varepsilon, y[1..j]) &= j \\
\mathrm{ed}(x[1..i], y[1..j]) &= \min(\mathrm{ed}(x[1..i-1], y[1..j-1]) + c, \\
&\qquad \mathrm{ed}(x[1..i-1], y[1..j]) + 1, \\
&\qquad \mathrm{ed}(x[1..i], y[1..j-1]) + 1)
\end{aligned}
$$

where $c = 0$ if $x[i] = y[j]$, otherwise $c = 1$.

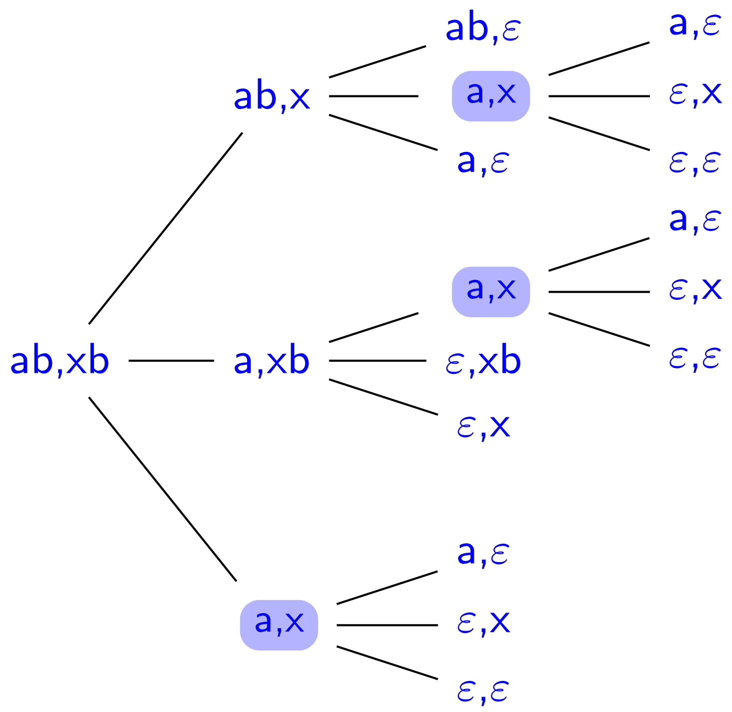
Brute Force Algorithm

ed-bf(x, y)

$m = |x|$, $n = |y|$
if $m = 0$ **then return** n
if $n = 0$ **then return** m
if $x[m] = y[n]$ **then** $c = 0$ **else** $c = 1$
return $\min(\text{ed-bf}(x, y[1 \ldots n-1]) + 1,$
$\quad\quad \text{ed-bf}(x[1 \ldots m-1], y) + 1,$
$\quad\quad \text{ed-bf}(x[1 \ldots m-1], y[1 \ldots n-1]) + c)$

Brute Force Algorithm

- Recursion tree for ed-bf(ab, xb):



- Exponential runtime in string length :-(
- Observation: Subproblems are computed repeatedly (e.g. ed-bf(a, x) is computed 3 times)
- Approach: Reuse previously computed results!

Outline

Dynamic Programming Algorithm – Top Down

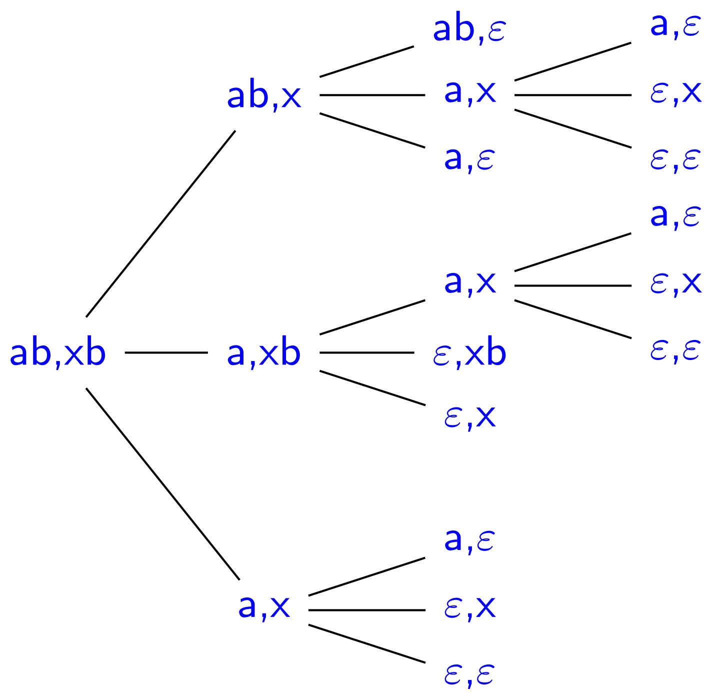
- Store distances between all prefixes of x and y
- Use matrix $C_{0..m,0..n}$ with

$$C_{i,j} = \text{ed}(x[1\ldots i], y[1\ldots j])$$

where $x[1..0] = y[1..0] = \varepsilon$.

- Example:

	ε	x	b
ε	0	1	2
a	1	1	2
b	2	2	**1**



Dynamic Programming Algorithm – Bottom Up

ed-dyn(x, y)

$C : array[0..|x|][0..|y|]$
for $i = 0$ **to** $|x|$ **do** $C[i, 0] = i$
for $j = 1$ **to** $|y|$ **do** $C[0, j] = j$
for $j = 1$ **to** $|y|$ **do**
    for $i = 1$ **to** $|x|$ **do**
        if $x[i] = y[j]$ **then** $c = 0$ **else** $c = 1$
        $C[i, j] = \min(C[i-1, j-1] + c,$
                    $C[i-1, j] + 1,$
                    $C[i, j-1] + 1)$

Dynamic Programming Algorithm – Properties

- Complexity:
  - $O(mn)$ time (nested for-loop)
  - $O(mn)$ space (the $(m+1) \times (n+1)$-matrix C)
- Improving space complexity (assume $m < n$):
  - we need only the previous column to compute the next column
  - we can forget all other columns
  - $\Rightarrow O(m)$ space complexity

Understanding the Solution

- Example:

x = `moon`
y = `mond`

ins →

	ren ↘	ε	m	o	n	d
	ε	0	←1	←2	←3	←4
del	m	↑1	↖0	←1	←2	←3
↓	o	↑2	↑1	↖0	←1	←2
	o	↑3	↑2	↖↑1	↖1	↖←2
	n	↑4	↑3	↑2	↖1	↖←2

- Each arrow represents an edit operation with minimal cost
- Cost 2 in cell (`n`, `d`) can either result from replacing `n` by `d` (diagonal arrow) or by inserting `d` (horizontal arrow)
- Each path from bottom right to top left corner represents a valid set of edit operations

Understanding the Solution

- Example:

x = `moon`
y = `mond`

ins →

del ↓

	ε	m	o	n	d
ε	0	1	2	3	4
m	1	0	1	2	3
o	2	1	0	1	2
o	3	2	1	1	2
n	4	3	2	1	2

```
m o o n
m o n d
```

```
m o o n
m   o n d
```

```
m o o n
m o   n d
```

- Solution 1: replace `n` by `d` and (second) `o` by `n` in x
- Solution 2: insert `d` after `n` and delete (first) `o` in x
- Solution 3: insert `d` after `n` and delete (second) `o` in x

Dynamic Programming Algorithm

ed-dyn$^+(x, y)$

$col_0 : array[0..|x|]$
$col_1 : array[0..|x|]$
for $i = 0$ **to** $|x|$ **do** $col_0[i] = i$
for $j = 1$ **to** $|y|$ **do**
  $col_1[0] = j$
  for $i = 1$ **to** $|x|$ **do**
    if $x[i] = y[j]$ **then** $c = 0$ **else** $c = 1$
    $col_1[i] = \min(col_0[i-1] + c,$
      $col_1[i-1] + 1,$
      $col_0[i] + 1)$
  $col_0 = col_1$

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Distance Metric

Definition (Distance Metric)

A distance function δ is a *distance metric* if and only if for any x, y, z the following hold:

- $\delta(x, y) = 0 \Leftrightarrow x = y$ (identity)
- $\delta(x, y) = \delta(y, x)$ (symmetric)
- $\delta(x, y) + \delta(y, z) \geq \delta(x, z)$ (triangle inequality)

Examples:

- the Euclidean distance is a metric
- $d(a, b) = a - b$ is not a metric (not symmetric)

Introducing Weights

- Look at the edit operations as a set of rules with a cost:

$$\begin{aligned}
\alpha(\varepsilon, b) &= \omega_{ins} && \text{(insert)} \\
\alpha(a, \varepsilon) &= \omega_{del} && \text{(delete)} \\
\alpha(a, b) &= \begin{cases} \omega_{rep} & \text{if } a \neq b \\ 0 & \text{if } a = b \end{cases} && \text{(replace)}
\end{aligned}$$

  where $a, b \in \Sigma$, and $\omega_{ins}, \omega_{del}, \omega_{rep} \in \mathbb{R}_0^+$.

- Edit script: sequence of rules that transform x to y
- Edit distance: edit script with minimum cost (adding up costs of single rules)
- Example: so far we assumed $\omega_{ins} = \omega_{del} = \omega_{rep} = 1$.

Weighted Edit Distance

- Recursive formula with weights:

$$
\begin{aligned}
C_{0,0} &= 0 \\
C_{i,j} &= \min(C_{i-1,j-1} + \alpha(x[i], y[j]), \\
&\qquad C_{i-1,j} + \alpha(x[i], \varepsilon), \\
&\qquad C_{i,j-1} + \alpha(\varepsilon, y[j]))
\end{aligned}
$$

  where $\alpha(a, a) = 0$ for all $a \in \Sigma$, and $C_{-1,j} = C_{i,-1} = \infty$.
- We can easily adapt the dynamic programming algorithm.

Variants of the Edit Distance

- Unit cost edit distance (what we did so far):
  - $\omega_{ins} = \omega_{del} = \omega_{rep} = 1$
  - $0 \leq ed(x, y) \leq \max(|x|, |y|)$
  - distance metric
- Hamming distance [Ham50, SK83]:
  - called also "string matching with k mismatches"
  - allows only replacements
  - $\omega_{rep} = 1$, $\omega_{ins} = \omega_{del} = \infty$
  - $0 \leq d(x, y) \leq |x|$ if $|x| = |y|$, otherwise $d(x, y) = \infty$
  - distance metric
- Longest Common Subsequence (LCS) distance [NW70, AG87]:
  - allows only insertions and deletions
  - $\omega_{ins} = \omega_{del} = 1$, $\omega_{rep} = \infty$
  - $0 \leq d(x, y) \leq |x| + |y|$
  - distance metric
  - $LCS(x, y) = (|x| + |y| - d(x, y))/2$

Allowing Transposition

- Transpositions
  - switch two adjacent characters
  - can be simulated by delete and insert
  - typos are often transpositions
- New rule for transposition

$$\alpha(ab, ba) = \omega_{trans}$$

allows us to assign a weight different from $\omega_{ins} + \omega_{del}$

- Recursive formula that includes transposition:

$$\begin{aligned} C_{0,0} &= 0 \\ C_{i,j} &= \min(C_{i-1,j-1} + \alpha(x[i], y[j]), \\ &\qquad C_{i-1,j} + \alpha(x[i], \varepsilon), \\ &\qquad C_{i,j-1} + \alpha(\varepsilon, y[j]), \\ &\qquad C_{i-2,j-2} + \alpha(x[i-1]x[i], y[j-1]y[j])) \end{aligned}$$

where $\alpha(ab, cd) = \infty$ if $a \neq d$ or $b \neq c$, $\alpha(a, a) = 0$ for all $a \in \Sigma$, and $C_{-1,j} = C_{i,-1} = C_{-2,j} = C_{i,-2} = \infty$.

Example: Edit Distance with Transposition

- Example: Compute distance between $x =$`meal` and $y =$`mael` using the edit distance with transposition ($\omega_{ins} = \omega_{del} = \omega_{rep} = \omega_{trans} = 1$)

	ε	m	a	e	l
ε	0	1	2	3	4
m	1	0	1	2	3
e	2	1	1	1	2
a	3	2	1	**1**	2
l	4	3	2	2	1

- The value in red results from the transposition of ea to ae.

Text Searching

- Goal:
  - search pattern p in text t ($|p| < |t|$)
  - allow k errors
  - match may start at any position of the text
- Difference to distance computation:
  - $C_{0,j} = 0$ (instead of $C_{0,j} = j$, as text may start at any position)
  - result: all $C_{m,j} \leq k$ are endpoints of matches

Example: Text Searching

- Example:

p = `survey`
t = `surgery`
k = 2

	ε	s	u	r	g	e	r	y
ε	0	0	0	0	0	0	0	0
s	1	0	1	1	1	1	1	1
u	2	1	0	1	2	2	2	2
r	3	2	1	0	1	2	2	3
v	4	3	2	1	1	2	3	3
e	5	4	3	2	2	1	2	3
y	6	5	4	3	3	**2**	**2**	**2**

- Solutions: 3 matching positions with $k \leq 2$ found.

```
s u r v e y
s u r g e

s u r v e y
s u r g e r

s u r v e   y
s u r g e r y
```

Summary

- Edit distance between two strings: the minimum number of edit operations that transforms one string into the another
- Dynamic programming algorithm with $O(mn)$ time and $O(m)$ space complexity, where $m \leq n$ are the string lengths.
- Basic algorithm can easily be extended in order to:
  - weight edit operations differently,
  - support transposition,
  - simulate Hamming distance and LCS,
  - search pattern in text with k errors.

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