

Similarity Search

The String Edit Distance

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Outline

- 1 String Edit Distance
 - Motivation and Definition
 - Brute Force Algorithm
 - Dynamic Programming Algorithm
 - Edit Distance Variants
- 2 Conclusion

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Motivation

- How different are
 - `hello` and `hello`?
 - `hello` and `hallo`?
 - `hello` and `hell`?
 - `hello` and `shell`?

What is a String Distance Function?

Definition (String Distance Function)

Given a finite alphabet Σ , a *string distance function*, δ_s , maps each pair of strings $(x, y) \in \Sigma^* \times \Sigma^*$ to a positive real number (including zero).

$$\delta_s : \Sigma^* \times \Sigma^* \rightarrow \mathbb{R}_0^+$$

- Σ^* is the set of all strings over Σ , including the *empty string* ε .

The String Edit Distance

Definition (String Edit Distance)

The *string edit distance* between two strings, $\text{ed}(x, y)$, is the minimum number of character insertions, deletions and replacements that transforms x to y .

- Example:
 - `hello` → `hallo`: replace `e` by `a`
 - `hello` → `hell`: delete `o`
 - `hello` → `shell`: delete `o`, insert `s`
- Also called *Levenshtein distance*.¹

¹Levenshtein introduced this distance for signal processing in 1965 [Lev65].

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Gap Representation

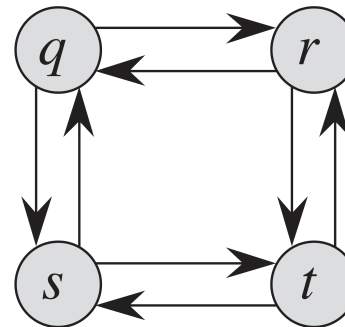
- **Gap representation** of the string transformation $x \rightarrow y$:
Place string x above string y
 - with a gap in x for every insertion,
 - with a gap in y for every deletion,
 - with different characters in x and y for every replacement.
- Any sequence of edit operations can be represented with gaps.
- **Example:**

```
    h a l l o  
s h e l l
```

- insert **s**
- replace **a** by **e**
- delete **o**

Deriving the Recursive Formula: Optimal Substructure

- **Recursive solution:** is applicable only to problems with optimal substructure property.
- **Optimal substructure** property of a problem:
 - optimal solution to larger problem computable from the optimal solutions of subproblems
- **Examples:**
 - Shortest path has optimal substructure: If a is on shortest path P from a to $b \Rightarrow$ the section $a \rightarrow c$ on P is the shortest path between a and c .
 - Longest simple² path does *not* have optimal substructure. Counter example [CLRS09]: Consider longest path $q \rightarrow t$ and subpath $q \rightarrow r$.



²i.e., the path has no cycles

Deriving the Recursive Formula: Optimal Substructure

Lemma (Optimal Substructure of String Edit Distance Problem)

Given a gap representation, $\text{gap}(x, y)$, between two strings x and y , such that the cost of $\text{gap}(x, y)$ is the string edit distance $\text{ed}(x, y)$.

If we remove the last column of $\text{gap}(x, y)$, then the gap representation of the remaining columns, $\text{gap}(x', y')$, has cost $\text{ed}(x', y')$ between the resulting substrings, x' and y' .

```
h a l l | o
s h e l l |
```

- **Example:**

- $x = \text{hallo}, y = \text{shell}, \text{cost}(\text{gap}(x, y)) = \text{ed}(x, y) = 3$
- $x' = \text{hall}, y = \text{shell}, \text{cost}(\text{gap}(x', y')) = \text{ed}(x', y') = 2$

Deriving the Recursive Formula: Optimal Substructure

Proof: Optimal Substructure String Edit Distance (by contradiction).

- Last column contributes with $c = 0$ or $c = 1$ to cost of $\text{gap}(x, y)$, thus:

$$\text{cost}(\text{gap}(x, y)) = \text{cost}(\text{gap}(x', y')) + c$$

- Assume $\text{gap}(x', y')$ is not optimal, i.e., $\text{cost}(\text{gap}(x', y')) > \text{ed}(x', y')$. Let $\text{gap}^*(x', y')$ be the respective gap representation:

$$\text{cost}(\text{gap}^*(x', y')) = \text{ed}(x', y') < \text{cost}(\text{gap}(x', y'))$$

- By extending $\text{gap}^*(x', y')$ with the last column, we get a gap representation $\text{gap}^*(x, y)$ with cost below $\text{ed}(x, y)$, which contradicts the definition of the edit distance.

$$\begin{aligned} \text{cost}(\text{gap}^*(x, y)) &= \text{cost}(\text{gap}^*(x', y')) + c \\ &< \text{cost}(\text{gap}(x', y')) + c = \text{ed}(x, y) \end{aligned}$$



Deriving the Recursive Formula

- Example:

h a l l o
s h e l l

- Notation:

- $x[1 \dots i]$ is the substring of the first i characters of x ($x[1 \dots 0] = \varepsilon$)
- $x[i]$ is the i -th character of x

- Recursive Formula:

$$\begin{aligned} \text{ed}(\varepsilon, \varepsilon) &= 0 \\ \text{ed}(x[1..i], \varepsilon) &= i \\ \text{ed}(\varepsilon, y[1..j]) &= j \\ \text{ed}(x[1..i], y[1..j]) &= \min(\text{ed}(x[1..i-1], y[1..j-1]) + c, \\ &\quad \text{ed}(x[1..i-1], y[1..j]) + 1, \\ &\quad \text{ed}(x[1..i], y[1..j-1]) + 1) \end{aligned}$$

where $c = 0$ if $x[i] = y[j]$, otherwise $c = 1$.

Brute Force Algorithm

ed-bf(x, y)

$m = |x|, n = |y|$

if $m = 0$ **then return** n

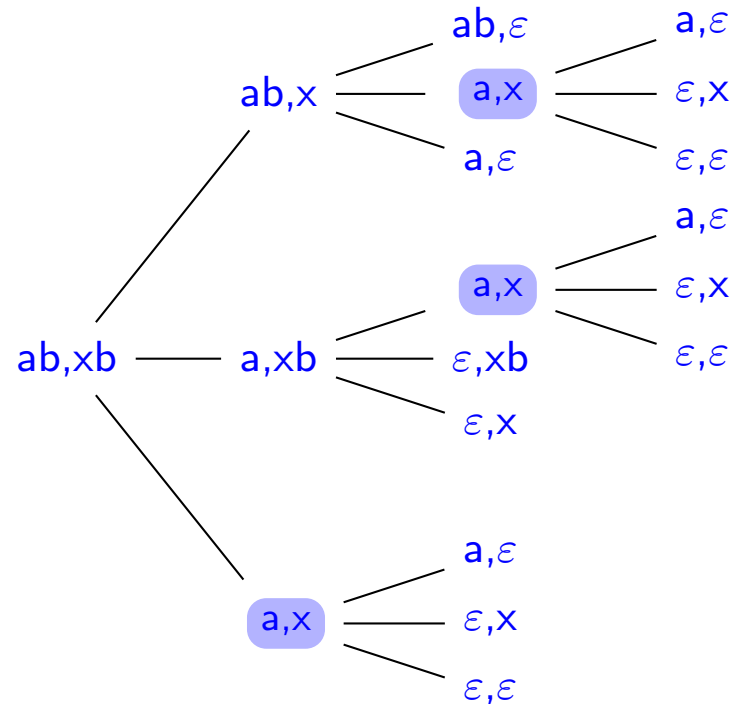
if $n = 0$ **then return** m

if $x[m] = y[n]$ **then** $c = 0$ **else** $c = 1$

return $\min(\text{ed-bf}(x, y[1 \dots n - 1]) + 1,$
 $\text{ed-bf}(x[1 \dots m - 1], y) + 1,$
 $\text{ed-bf}(x[1 \dots m - 1], y[1 \dots n - 1]) + c)$

Brute Force Algorithm

- Recursion tree for $\text{ed-bf}(\text{ab}, \text{xb})$:



- Exponential runtime in string length :- (
- **Observation:** Subproblems are computed repeatedly (e.g. $\text{ed-bf}(\text{a}, \text{x})$ is computed 3 times)
- **Approach:** Reuse previously computed results!

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Dynamic Programming Algorithm – Top Down

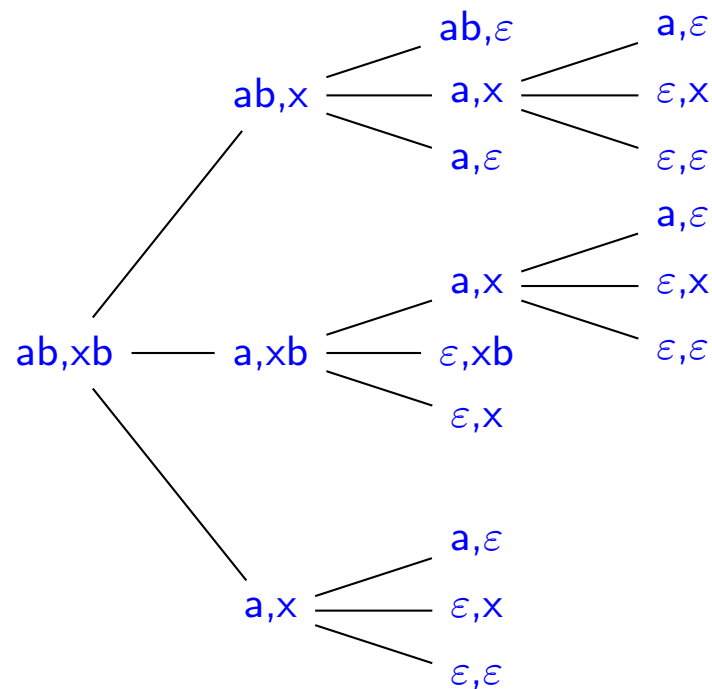
- Store distances between all prefixes of x and y
- Use matrix $C_{0..m,0..n}$ with

$$C_{i,j} = \text{ed}(x[1 \dots i], y[1 \dots j])$$

where $x[1..0] = y[1..0] = \varepsilon$.

- Example:

	ε	x	b
ε	0	1	2
a	1	1	2
b	2	2	1



Dynamic Programming Algorithm – Bottom Up

ed-dyn(x, y)

```
 $C$  : array[0.. $|x|$ ][0.. $|y|$ ]  
for  $i = 0$  to  $|x|$  do  $C[i, 0] = i$   
for  $j = 1$  to  $|y|$  do  $C[0, j] = j$   
for  $j = 1$  to  $|y|$  do  
    for  $i = 1$  to  $|x|$  do  
        if  $x[i] = y[j]$  then  $c = 0$  else  $c = 1$   
         $C[i, j] = \min(C[i - 1, j - 1] + c,$   
                      $C[i - 1, j] + 1,$   
                      $C[i, j - 1] + 1)$ 
```

Dynamic Programming Algorithm – Properties

- Complexity:
 - $O(mn)$ time (nested for-loop)
 - $O(mn)$ space (the $(m+1) \times (n+1)$ -matrix C)
- Improving space complexity (assume $m < n$):
 - we need only the previous column to compute the next column
 - we can forget all other columns
 - $\Rightarrow O(m)$ space complexity

Understanding the Solution

- Example:

$x = \text{moon}$
 $y = \text{mond}$

		ins →				
	ren ↘	ϵ	m	o	n	d
	ϵ	0	←1	←2	←3	←4
	del	↑1	0	←1	←2	←3
	↓	↑2	↑1	0	←1	←2
		↑3	↑2	↑1	1	←2
		↑4	↑3	↑2	↑1	←2
	n					

- Each arrow represents an edit operation with minimal cost
- Cost 2 in cell (n , d) can either result from replacing n by d (diagonal arrow) or by inserting d (horizontal arrow)
- Each path from bottom right to top left corner represents a valid set of edit operations

Understanding the Solution

- Example:

$x = \text{moon}$
 $y = \text{mond}$

		ins →					
		ϵ	m	o	n	d	
	ϵ	0	1	2	3	4	
del ↓	m	1	0	1	2	3	m o o n
	o	2	1	0	1	2	m o n d
	o	3	2	1	1	2	
	n	4	3	2	1	2	m o o n
							m o n d

- Solution 1:** replace **n** by **d** and (second) **o** by **n** in x
- Solution 2:** insert **d** after **n** and delete (first) **o** in x
- Solution 3:** insert **d** after **n** and delete (second) **o** in x

Dynamic Programming Algorithm

$\text{ed-dyn}^+(x, y)$

```
col0 : array[0..|x|]  
col1 : array[0..|x|]  
for i = 0 to |x| do col0[i] = i  
for j = 1 to |y| do  
    col1[0] = j  
    for i = 1 to |x| do  
        if x[i] = y[j] then c = 0 else c = 1  
        col1[i] = min(col0[i - 1] + c,  
                     col1[i - 1] + 1,  
                     col0[i] + 1)  
  
col0 = col1
```

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Distance Metric

Definition (Distance Metric)

A distance function δ is a *distance metric* if and only if for any x, y, z the following hold:

- $\delta(x, y) = 0 \Leftrightarrow x = y$ (identity)
- $\delta(x, y) = \delta(y, x)$ (symmetric)
- $\delta(x, y) + \delta(y, z) \geq \delta(x, z)$ (triangle inequality)

Examples:

- the Euclidean distance is a metric
- $d(a, b) = a - b$ is not a metric (not symmetric)

Introducing Weights

- Look at the **edit operations** as a set of **rules with a cost**:

$$\begin{aligned}\alpha(\varepsilon, b) &= \omega_{ins} && \text{(insert)} \\ \alpha(a, \varepsilon) &= \omega_{del} && \text{(delete)} \\ \alpha(a, b) &= \begin{cases} \omega_{rep} & \text{if } a \neq b \\ 0 & \text{if } a = b \end{cases} && \text{(replace)}\end{aligned}$$

where $a, b \in \Sigma$, and $\omega_{ins}, \omega_{del}, \omega_{rep} \in \mathbb{R}_0^+$.

- Edit script**: sequence of rules that transform x to y
- Edit distance**: edit script with minimum cost
(adding up costs of single rules)
- Example**: so far we assumed $\omega_{ins} = \omega_{del} = \omega_{rep} = 1$.

Weighted Edit Distance

- Recursive formula with weights:

$$\begin{aligned}C_{0,0} &= 0 \\C_{i,j} &= \min(C_{i-1,j-1} + \alpha(x[i], y[j]), \\&\quad C_{i-1,j} + \alpha(x[i], \varepsilon), \\&\quad C_{i,j-1} + \alpha(\varepsilon, y[j]))\end{aligned}$$

where $\alpha(a, a) = 0$ for all $a \in \Sigma$, and $C_{-1,j} = C_{i,-1} = \infty$.

- We can easily adapt the dynamic programming algorithm.

Variants of the Edit Distance

- **Unit cost** edit distance (what we did so far):
 - $\omega_{ins} = \omega_{del} = \omega_{rep} = 1$
 - $0 \leq ed(x, y) \leq \max(|x|, |y|)$
 - distance metric
- **Hamming** distance [Ham50, SK83]:
 - called also “string matching with k mismatches”
 - allows only replacements
 - $\omega_{rep} = 1, \omega_{ins} = \omega_{del} = \infty$
 - $0 \leq d(x, y) \leq |x|$ if $|x| = |y|$, otherwise $d(x, y) = \infty$
 - distance metric
- **Longest Common Subsequence (LCS)** distance [NW70, AG87]:
 - allows only insertions and deletions
 - $\omega_{ins} = \omega_{del} = 1, \omega_{rep} = \infty$
 - $0 \leq d(x, y) \leq |x| + |y|$
 - distance metric
 - $LCS(x, y) = (|x| + |y| - d(x, y))/2$

Allowing Transposition

- Transpositions
 - switch two adjacent characters
 - can be simulated by delete and insert
 - typos are often transpositions
- New rule for transposition

$$\alpha(ab, ba) = \omega_{trans}$$

allows us to assign a weight different from $\omega_{ins} + \omega_{del}$

- Recursive formula that includes transposition:

$$\begin{aligned} C_{0,0} &= 0 \\ C_{i,j} &= \min(C_{i-1,j-1} + \alpha(x[i], y[j]), \\ &\quad C_{i-1,j} + \alpha(x[i], \varepsilon), \\ &\quad C_{i,j-1} + \alpha(\varepsilon, y[j]), \\ &\quad C_{i-2,j-2} + \alpha(x[i-1]x[i], y[j-1]y[j])) \end{aligned}$$

where $\alpha(ab, cd) = \infty$ if $a \neq d$ or $b \neq c$, $\alpha(a, a) = 0$ for all $a \in \Sigma$,
and $C_{-1,j} = C_{i,-1} = C_{-2,j} = C_{i,-2} = \infty$.

Example: Edit Distance with Transposition

- **Example:** Compute distance between $x = \text{meal}$ and $y = \text{mael}$ using the edit distance with transposition ($\omega_{ins} = \omega_{del} = \omega_{rep} = \omega_{trans} = 1$)

	ε	m	a	e	l
ε	0	1	2	3	4
m	1	0	1	2	3
e	2	1	1	1	2
a	3	2	1	1	2
l	4	3	2	2	1

- The value in red results from the transposition of ea to ae.

Text Searching

- Goal:
 - search pattern p in text t ($|p| < |t|$)
 - allow k errors
 - match may start at any position of the text
- Difference to distance computation:
 - $C_{0,j} = 0$ (instead of $C_{0,j} = j$, as text may start at any position)
 - result: all $C_{m,j} \leq k$ are endpoints of matches

Example: Text Searching

- Example:

p = survey
 t = surgery
 k = 2

	ε	s	u	r	g	e	r	y
ε	0	0	0	0	0	0	0	0
s	1	0	1	1	1	1	1	1
u	2	1	0	1	2	2	2	2
r	3	2	1	0	1	2	2	3
v	4	3	2	1	1	2	3	3
e	5	4	3	2	2	1	2	3
y	6	5	4	3	3	2	2	2

- Solutions: 3 matching positions with $k \leq 2$ found.

s u r v e y
 s u r g e

s u r v e y
 s u r g e r

s u r v e y
 s u r g e r y

Summary

- **Edit distance** between two strings: the minimum number of edit operations that transforms one string into the another
- **Dynamic programming algorithm** with $O(mn)$ time and $O(m)$ space complexity, where $m \leq n$ are the string lengths.
- Basic algorithm can easily be **extended** in order to:
 - weight edit operations differently,
 - support transposition,
 - simulate Hamming distance and LCS,
 - search pattern in text with k errors.



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